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SOME DEGENERATE AND GENERALIZED WAVE MODELS IN ELASTO- AND HYDRODYNAMICS†

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Models, degenerate with respect to a small parameter, which describe the propagation of disturbances with a finite velocity, are considered. Example, when parabolic models degenerate into hyperbolic ones and, inversely, when hyperbolic models can be obtained by a generalization of parabolic models, are given. The reduction of the three-dimensional problem of elasto-dynamics for a layer to a two-dimensional one on the basis of power series, and also the construction of weakly dispersive, but strongly non-linear models of the propagation of surface waves on water, close to hyperbolic models, are considered in more detail.

The problem of the finiteness of the velocity of propagation of disturbances as a condition for the correctness of the formulation of the initial boundary-value problem for hyperbolic equations and its solvability were considered by Hersh in [1]. He presented some cases of the non-existence of solutions for initial boundary value problems with finite velocities. He also examined correctly formulated initial boundary-value problems for hyperbolic equations in a semi-infinite domain and defined the conditions for a finite velocity. These problems have also been considered by other researchers ([2–9], etc.).

The analysis of dynamic models from the viewpoint of the finiteness of the velocity of propagation of disturbances involving a small parameter β, when the original models degenerate into simplified models (quasi-degenerate or degenerate with respect to parameter β) for $\beta \to 0$, is of interest in wave theory. The necessary condition of the finiteness of velocity of propagation of disturbances is the hyperbolity of the model, i.e. it has to be described by a hyperbolic system of differential equations. It should be noted that the type of equation may change for asymptotic degeneration. Therefore, only limit hyperbolic models obtained in this case are of interest, for instance, when the parabolic model transforms into the hyperbolic one or the hyperbolic model of elasto-dynamics for a layer transforms into the hyperbolic model, and not into the classic parabolic model, which predicts an infinite velocity of propagation of disturbances.

These models transform into hyperbolic models of a lower order for a singular degeneration in the case of dissipative models described by parabolic equations, which predict an infinite velocity of propagation of disturbances.

On the other hand, dissipative and diffusion parabolic models can be generalized to hyperbolic ones by an expansion (complement) of the parabolic operator up to the hyperbolic one on the basis of the generalized transport equation (an idea originated by Maxwell [9]). The solutions of the parabolic equations will then be the limit solutions of these generalized equations [10, 11].

The three-dimensional problem can be reduced to a two-dimensional one in the case of the degeneration of the original hyperbolic model of elasto-dynamics for a layer with a small relative thickness using the power series method without invoking additional hypotheses, introduced by employing traditional phenomenological approaches. This results in a set of simplified (degenerate) models of various types, from which only hyperbolic models are selected by the proposed algorithm. Consequently, the hyperbolic model of flexural vibrations of Timoshenko–Mindlin plates (the two-mode approximation) and a more accurate model (the three-mode approximation) can be obtained analytically. These models describe the propagation of disturbances with a finite velocity, unlike the classical models. An accurate formula for determining the shear coefficient, introduced artificially in all traditional approaches when constructing refined equations for the vibration of rods, plates and shells, in the Timoshenko–Mindlin model can also be obtained on this basis, and the choice of this coefficient has been subject of many studies and discussions.

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In the case of a fluid of finite varying depth, it is possible to construct quasi-degenerate models for small dispersion β and large non-linearity α, which leads to models close to hyperbolic ($\beta = 0$), using an analytical approach. Unlike traditional approaches, the limitation on the non-linearity parameter α is removed here, and the system of evolutional equations thereby obtain describes the propagation of highly non-linear waves. Consequently, a consideration of such non-linear effects and the variability of depth results in a perturbation of the propagating solitary wave and the formation of "tails" as opposed to a purely soliton wave for $\alpha \sim \beta$ or tipping waves for $\beta = 0$.

The singular degeneration of parabolic models to hyperbolic models with respect to a parameter. A typical example is the Navier–Stokes equation for a compressible when the viscosity approaches zero. Other examples include the Burgers equation $u_t + uu_x = \gamma u_{xx}$ for $\gamma \to 0$ and the Korteweg–de Vries equation $u_t + \alpha uu_x + \beta u_{xxx} = 0$ for $\beta \to 0$ as well as the degeneration of hyperbolic-parabolic models of magnets hydrodynamics and magneto-elasticity into hyperbolic ones as the magnetic Reynolds number approaches infinity [12]. These and many other evolutionary equations describe wave propagation due to the fact they include a hyperbolic operator of lower order as a kernel [13]. The non-dissipative Korteweg–de-Vries equation also degenerates into a hyperbolic equation but in addition it predicts the propagation of solitons with a finite velocity for a balance of non-linear and dispersion effects.

Hyperbolic models as a generalization of parabolic models. Examples of the generalization (expansion) of parabolic equations, describing the propagation of disturbances with an infinite velocity, into hyperbolic equations were given in [5–14]. We will also note the model of heat conduction, the diffusion model, the Smoluchowski equation, the model of the evolution of floor sediments [13, 14], the model of the elasticity of a relativistic body and the model of turbulence. One of the most characteristic models is the hyperbolic model [14], which generalizes the Navier–Stokes equation by expansion of the parabolic operator to a hyperbolic one. The hyperbolic generalization was presented in the form of a transport equation, predicting the existence of a front propagating with a finite velocity in a solution of the travelling waves type. This is not predicted by the well-known reaction-diffusion type equations.

The construction of improved wave models of the theory of plates. Usually the equations of longitudinal and bending vibrations of plates, including also the well-known improved equations, are derived by invoking physical and geometrical hypotheses. These equations are essentially approximate constructions (approximations) of the problem of elasto-dynamics for an elastic layer in an exact formulation. Here the approximate equations are derived from this exact formulation, without invoking my hypotheses of a physical nature and on the basis of the analytical algorithm of the reduction of the three-dimensional problem to a two-dimensional one on the assumption that the thickness of the layer is small compared with the horizontal scale. Such an algorithm can be implemented either by the power-series method or asymptotic expansion methods. This obviously results in infinite systems, the reduction of which results in a set of approximate models, which can be seen as degenerate (quasi-degenerate) with respect to the small transverse coordinate or a small parameter. Thus, all known phenomenological models and new more accurate (generalized) models, which cannot be practically constructed on the basis of hypotheses of a physical and geometrical nature, can be obtained analytically. Our aim is to select from the set of approximations only those approximations which result in hyperbolic models describing the propagation of disturbances with a finite velocity.

We will consider the problem of elasto-dynamics for an infinite layer of thickness $2h$, bounded by end surfaces $x_3 = \pm h$ in the region

$$\Omega = \left\{(x_1, x_2, x_3) \in R^3 : x_1, x_2 \in (-\infty, \infty), x_3 \in \left[-\frac{\xi}{2}, \frac{\xi}{2}\right]\right\}; \quad \xi = -\frac{2h}{l} \tag{1}$$

in a rectangular Cartesian system of coordinates (x_1, x_2, x_3) where l is the horizontal scale of the layer. The initial boundary-value problem for the displacement vector $\mathbf{u} = (u_1, u_2, u_3 = w)$ is formulated as follows: it is required to find the vector-function $\mathbf{u} = \mathbf{u}\,(x_1, x_2, x_3, t)$ as a solution of the hyperbolic equations in the domain $\Omega \times [0, T]$, $T > 0$

$$\nabla^2 u_k + (1 + \lambda/G)\partial_k(\nabla \cdot \mathbf{u}) = \partial_{tt} u_k, \quad k = 1, 2, 3 \tag{2}$$

which satisfy the boundary conditions for the components of the stress tensor on the end surfaces of the layer

$$x_3 = \pm\xi/2: \quad \sigma_{33} = q^{\pm}(x_1, x_2, t), \quad \sigma_{3i} = p_i^{\pm}(x_1, x_2, t), \quad i = 1, 2 \tag{3}$$

and the initial conditions

$$t = 0: \quad u_k = 0, \quad \partial_t u_k = 0, \quad k = 1, 2, 3 \tag{4}$$

Dimensionless quantities are introduced everywhere. The characteristic length l, the shear modulus G and the velocity of propagation of shear waves c_s are adopted as scales. It is assumed that λ and G are constant quantities.

Henceforth we will assume that the thickness of the layer is small $\xi \ll 1$, and thus it is natural to use an expansion in the dimensionless coordinate x_3 relative to the median surface $x_3 = 0$, thereby reducing the dimension of the problem [15–17]. This results in degeneration of the original hyperbolic model, where three cases of degeneration are possible, resulting in equations of the parabolic, hyperbolic and mixed type. Only the degeneration of the hyperbolic model into a hyperbolic model is correct and has a physical meaning for the condition that the velocity of propagation of the perturbations is finite.

The required functions are represented in the form of a power series

$$u_k(x_1, x_2, x_3, t) = \sum_{m=0}^{\infty} u_{km}(x_1, x_2, t)x_3^m, \quad k = 1, 2, 3$$

As a result the original problem (1)–(4) reduces to determining an infinite number of functions u_{km}, which satisfy an infinite system of differential equations and recurrence relations. In turn, this infinite system decomposes into two independent subsystems, corresponding to symmetric (planar) and asymmetric (bending) deformations about the median surface $x_3 = 0$. Here we will only consider the case of asymmetric deformations, for which we obtain, after lengths reduction, (the summation is carried out from $s = 0$ to $s = \infty$)

$$\begin{aligned}
&e(x_1, x_2, x_3, t) = \sum \tilde{e}_{2s+1}(x_1, x_2, t)x_3^{2s+1}, \quad e = u_{i,i}, \quad i = 1, 2 \\
&w(x_1, x_2, x_3, t) = \sum \tilde{w}_{2s}(x_1, x_2, t)x_3^{2s} \\
&\sum[(2s+1)\tilde{e}_{2s+1} + \nabla^2\tilde{w}_{2s}]2^{-2s}\xi^{2s} = \frac{\partial}{\partial x_1}\frac{1}{2}(p_1^+ + p_1^-) + \frac{\partial}{\partial x_2}\frac{1}{2}(p_2^+ + p_2^-) \\
&\sum\left[-\tilde{e}_{2s+1} - \frac{1}{2s+1}L_s\tilde{w}_{2s}\right]2^{-(2s+1)}\xi^{2s+1} = \frac{1}{2}(q^+ - q^-) \\
&\tilde{w}_{2s+2} = -\frac{1+\lambda/G}{(2s+2)(2+\lambda/G)}\tilde{e}_{2s+1} - \frac{1}{(2s+1)(2s+2)(2+\lambda/G)}L_s\tilde{w}_{2s} \\
&\tilde{e}_{2s+3} = \frac{1}{(2s+2)(2s+3)}\left[-L_e + \frac{1+\lambda/G}{2+\lambda/G}\nabla^2\right]\tilde{e}_{2s+1} + \\
&+ \frac{1+\lambda/G}{(2s+1)(2s+2)(2s+3)(2+\lambda/G)}\nabla^2 + L_s w_{2s}
\end{aligned} \tag{5}$$

after where

$$L_s = c_s^2\nabla^2 - \frac{\partial^2}{\partial t^2}, \quad L_e = c_e^2\nabla^2 - \frac{\partial^2}{\partial t^2}$$

e is the divergence of the planar displacements, w is the deflection, $p_i^{\pm}$ and $q^{\pm}$ are the shear and normal loads along the end surfaces of the layers and c_s and c_e are the velocities of propagation of shear and dilatation waves.

Equation (5) gives the exact solution of the problem. Reduction of this system enables us to obtain a set of approximations of various kinds. The hyperbolic degeneration of the initial boundary-value problem for the final hyperbolic system of equations of arbitrary order in R^n on the basis of the power series method was considered in [17]. Thus, necessary and sufficient conditions of the following degeneration: the completeness of the reduced system and the conservation of all space-time differential operations up to a certain order, were established.

The truncation of Eq. (5) to the seventh order inclusive results in a three-mode (thickness wave modes) approximation, which can be reduced to the following equation

$$\left\{\left[\left(\xi\frac{\partial^2}{\partial t^2}+\xi^3 a_1\nabla^2\nabla^2\right)_K-\xi^3 a_2\frac{\partial^2}{\partial t^2}\nabla^2+\xi^3 a_3\frac{\partial^4}{\partial t^4}\right]_{TM}-\xi^5 b_1\nabla^2\nabla^2\nabla^2+\right.$$
$$\left.+\xi^5 b_2\frac{\partial^2}{\partial t^2}\nabla^2\nabla^2-\xi^5 b_3\frac{\partial^4}{\partial t^4}\nabla^2+\xi^5 b_4\frac{\partial^6}{\partial t^6}\right\}_{TMS} w=\left\{\left[1-\xi^2 d_1\nabla^2+\xi^2 d_2\frac{\partial^2}{\partial t^2}\right]_{TM}+\right. \tag{6}$$
$$\left.+\xi^4 d_3\nabla^2\nabla^2-\xi^4 d_4\frac{\partial^2}{\partial t^2}\nabla^2+\xi^4 d_5\frac{\partial^4}{\partial t^4}\right\}_{TMS}(q^+-q^-)$$

Timoshenko [18] generalized the parabolic Bernoulli–Euler model of the flexural vibrations of a beam to a hyperbolic model on a phenomenological basis by introducing corrections responsible for the thickness-shear deformations and the inertia of rotation. On this basis Mindlin [19] generalized the parabolic Kirchhoff model of the flexural vibration of plates [20] (the operator K in Eq. (6)) to a hyperbolic model (the two-mode model – TM operators). A more general hyperbolic model [21] was constructed as a mathematical approximation without introducing any phenomenological assumptions (the three-mode approximation – TMS operators) including also the two-dimensional system as a special case. It should be noted that the coefficients a_p, b_q and d_r in Eq. (6) depend only on Poisson's ratio ν. This enables the exact magnitude of the shear coefficient in the relation $k^2 = 2/(2-\nu+\sqrt{1/2+\nu^2})$ to be determined from Eq. (6).

A development of the theory of constructing refined models of the dynamics of rods, plates and shells has been given and their multiple applications have been described in [22].

The model of non-linear wave propagation in water, degenerate with respect to the dispersion parameter. The same approach as in the case of the vibration of plates is used here to solve the problem. The problem of the propagation of surface gravitational waves, which in the majority of cases is well described by the model of an ideal incompressible fluid for its potential motion, is considered. As a result the determination of the vector field reduces to the scalar problem for the velocity potential φ and the deviation of the free surface η. The problem is considered in the complete non-linear form for a fluid of varying depth with an undisturbed free surface $z = 0$ in a rectangular Cartesian system of coordinates x, y, z.

Below, the plane problem is considered, i.e. the solutions are independent of the coordinate y. The problem is characterised by the three determining dimensionless parameters

$$\alpha = a/H_0,\quad \beta = (H_0/l)^2,\quad \gamma = \mathrm{tg}\theta = H_0/l,\quad \mathrm{Ur} = \alpha/\beta$$

where θ is the angle of deflection of the floor, Ur is the Ursell number (a derived parameter), H_0 is the depth (the vertical scale), l is the characteristic horizontal scale and a the maximum deflection of the free surface (the amplitude). The problem is considered in domain

$$\Omega = \{(x, y, z) \in R^3 \,|\, \tilde{x} \le x < \infty,\ -\infty < y < \infty,\ -H(x) \le z \le \alpha\eta(x, t)\} \tag{7}$$

where $x = \tilde{x}$ is the line in front of the zone of wave erosion.

In dimensionless variables

$$x^* = \frac{x}{l},\quad z^* = \frac{z}{H_0},\quad t^* = \frac{c_0}{l}t = \frac{\sqrt{gH_0}}{l}t,\quad \varphi^* = \frac{c_0}{gla}\varphi,\quad \eta^* = \frac{\eta}{a}$$

the problem is formulated in terms of two unknown functions φ and η in the following form (the asterix is omitted below)

$$\beta\varphi_{xx} + \varphi_{zz} = 0 \text{ in domain } \Omega \tag{8}$$

$$z = -H(x)\colon\ \varphi_z + \beta H_x\varphi_x = 0 \tag{9}$$

$$z = \alpha\eta:\ \eta_t + \alpha\eta_x\varphi_x - \beta^{-1}\varphi_z = 0, \quad \eta + \varphi_t + (\alpha/2)\varphi_x^2 + (\alpha/(2\beta))\varphi_z^2 = 0 \tag{10}$$

$$t = 0:\ \varphi(x, z, t) = f_1(x, z), \quad \varphi_t(x, z, t) = f_2(x, z) \tag{11}$$

It should be noted that not one (sufficient) parameter but three scaling parameters H_0, l, a have been introduced, which is necessary for the asymptotic analysis.

The solution of problem (8)–(11) for the case of propagating waves in a complete non-linear formulation is unknown. This problem describes a non-linear-dispersion system, for which the propagation of solitary waves is typical. Here an approximate analysis is given using an asymptotic method [23, 24], which enables the problem to be reduced to an analysis of a system of two evolutionary equations. It is assumed that the dispersion parameter β and the gradient of the floor surface γ are small and that at the same time the non-linear parameter α is considered to be arbitrary, unlike the traditional approaches in which it is assumed that $\alpha \sim \beta$.

The two-dimensional problem with respect to the coordinates x, z is reduced to a one-dimensional problem with respect to x using the power series method, which reduces the problem to an infinite system, including the terms β^q and α^n (q, n are finite integer values) and their products. The introduction of assumptions about the smallness of the parameters β and γ enables the infinite systems to be reduced which retaining terms of the order of β, β^2, ... , which corresponds to the long-wave approximations. Further, we are retaining in the infinite system terms of the first order only with respect to β and all terms with the non-linear parameter α^n. This corresponds to the strongly non-linear weakly dispersion model, degenerate with respect to the dispersion parameter β [25]. The opposite limiting case $\beta \gg \alpha$ results in parabolic models when dispersion effects are taken into account completely, while non-linear effects are small and are not considered here.

We will represent the function φ in the form of the expansion

$$\varphi(x, z, t) = \sum_{n=0}^{\infty} (z + H)^n \beta^n f^{(n)}(x, t) \tag{12}$$

It can be seen that expansions in the parameter β and $z + H$ are equivalent.

We substitute Eq. (12) into Eqs (8)–(11). The substitution into Eq. (8) results in the recurrent relation

$$\frac{1}{\beta} f_{xx}^{(k)} = 2(k+1)H_x f_x^{(k+1)} + (k+1)H_{xx} f^{(k+1)}$$

The condition on the bottom (9) enables us to express $f^{(1)}$ in terms of $f^{(0)} = f$

$$f^{(1)} = -H_x f^0 - \beta H_x^3 f^0 + O(\beta^2)$$

The final expression for φ has the form

$$\begin{aligned} \varphi &= f - \beta\left[\left((z+H)H_x f_x + \frac{(z+H)^2}{2} f_{xx}\right)\right] + \\ &+ \beta^2\left[-H_x^3 f_x (z+H) + \frac{3}{2}(z+H)^2 (H_x H_{xx} f_x + H_x^2 f_{xx}) = \right. \\ &= \left.\frac{(z+H)^3}{2}\left(\frac{1}{3} H_{xxx} f_x + \ldots + \frac{(z+H)^4}{24} f_{xxxx}\right)\right] + O(\beta^3) \end{aligned} \tag{13}$$

We obtain the equation

$$\begin{aligned} &\eta_t + h_x\omega + h\omega_x - \beta\left[h_x\left(\frac{3}{2}H_{xx}\omega_x + \frac{3}{2}H_x\omega_{xx} + \frac{1}{2}H_{xxx}\omega + \frac{\alpha}{2}\eta_x\omega_{xx}\right) + \right. \\ &\left. + h(\alpha\eta_x H_{xx}\omega + 3H_x^2\omega_x + 2\alpha\eta_x H_x\omega_x + 3H_x H_{xx}\omega) + \frac{1}{6}h^3\omega_{xxx} + \alpha\eta_x H_x^2\omega)\right] = O(\beta^2) \end{aligned} \tag{14}$$

by substituting Eq. (13) into the first boundary condition (10), retaining the terms of the order of β and $\alpha^n\beta$ and taking into account that $h = H + \alpha\eta$, $\omega = f_x$.

The second condition of (10) reduces to the following equation

$$\omega_t + \eta_x + \alpha\frac{1}{2}(\omega_x^2)_x - \beta\Big\{\frac{1}{2}h^2[\omega_{xt} + \alpha(\omega\omega_{xx} - \omega_x^2)] + \omega_t(hH_x)_x + \omega_{xt}hH_x +$$
$$+ \alpha\omega\omega_x(h_xH_x + H_x^2 + 3hH_{xx}) + \tag{15}$$
$$+ \alpha hH_x\omega_x^2 + \alpha\omega\omega_{xx}hH_x + \omega^2[H_{xx}(h_x + H_x) + hH_{xxx}]\Big\}_x = O(\beta^2)$$

after differentiating it with respect to x and substituting expression (13) into it.

The evolution equations (14) and (15) form a closed system of coupled equations independent of z.

Below, we introduce the average velocity (averaged over the depth)

$$u = \frac{1}{h}\int_{-H}^{\alpha\eta}\phi_z dz = \omega - \beta\Big[\frac{1}{2}hH_{xx}\omega + hH_x\omega_x + H_x^2\omega + \frac{1}{6}H^2\omega_{xx}\Big] + O(\beta^2) \tag{16}$$

The evolution equations (14) and (15) take the form

$$\eta_t + (hu)_x = 0 \tag{17}$$

$$u_t + \alpha uu_x + \eta_x = \beta\Big(\frac{H^3}{3}u_{xxt} + HH_xu_{xt} + \frac{H}{2}H_{xx}u_t\Big) +$$
$$+ \alpha\beta\Big[(\eta H)_xu_{xt} + HH_xuu_{xx} + \frac{2}{3}\eta Hu_{xxt} + \frac{H^2}{3}uu_{xxx} - \frac{H^2}{3}u_xu_{xx} + \frac{H}{2}H_{xx}u_t + \tag{18}$$
$$+ \frac{3}{2}HH_{xx}uu_x + \frac{H}{2}H_{xxx}u^2 + \eta_xH_xu_t\Big] + L_1 + O(\beta^2)$$

after some reduction using expression (16), where L_1 is an operator taking into account higher-order non-linearity, i.e. $O(\alpha^2\beta, \alpha^3\beta, \alpha^4\beta)$.

System of equations (17) and (18) reduces to the well-known equations [26]

$$\eta_t + (hu)_x = 0$$
$$u_t + \alpha uu_x + \eta_x = \beta\Big(\frac{H^3}{3}u_{xxt} + HH_xu_{xt} + \frac{H}{2}H_{xx}u_t\Big) + O(\beta^2)$$

in the case when the non-linearity parameter α is small and is of the same order of magnitude as the dispersion parameter β, $\alpha \sim \beta \ll 1$.

When $\beta = 0$ system (17), (18) reduces to the system of quasi-linear equations for waves in shallow water

$$u_t + \alpha uu_x + \eta_x = 0, \quad \eta_t + (hu)_x = 0$$

from which follow the linearized equations

$$u_t = -\eta_x, \quad \eta_t = -(Hu)_x$$

when $\alpha = 0$, and they reduce to the wave equation

$$\frac{\partial}{\partial x}\Big(H\frac{\partial\varphi}{\partial x}\Big) - \frac{\partial^2\varphi}{\partial t^2} = 0$$

when $u = \partial\varphi/\partial x$.

The evolution equations (17) and (18) are of interest for analysing wave propagation in the coastal zone, where the characteristic distances are relatively small, so that dispersion effects do not accumulate, while at the same time non-linear effects are significant. The analysis of the initial boundary-value problem using of Eqs (17) and (18), which describe the transformation of solitary waves at the point where they hit the shore, demonstrates that large non-linear effects and the variability of the depth during the propagation of solitons is accompanied by a distortion of the wave profile and the occurrence of oscillating tails [25].

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